CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar,
Friday conversations:
For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at “free formulas” and their combinatorics
2. How to construct free objects
   1. w.r.t. a functor with - at least - two combinatorial applications:
      1. the two routes to reach the free algebra
      2. alphabets interpolating between commutative and non commutative worlds
   2. without functor: sums, tensor and free products
   3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

CCRT[18] Initial topologies, Dom(Li) and Stars of the Plane.

1. Iterated integrals (trees, $S' = MS$, primitives, sectioned subalgebras)
2. Convergence of Picard's process (initial topologies, standard topology of $\mathcal{H}(\Omega)$, ultrametric and Treves topologies, paths drawn on the Magnus group)
3. The arrow Li ($Dom(\mathrm{Li})$)
4. Integrators ι_i (discontinuity of ι_0)
5. Discussion, open questions
    1. Topological complexity of $Dom(\mathrm{Li}), \mathrm{Li}(Dom(\mathrm{Li}))$
    2. Baire class of ι_0
6. Some concluding remarks.

$S' = MS$ and iterated integrals.

1. Let (R, ∂) be a differential ring and X an alphabet. Then immediately $(R\langle\langle X\rangle\rangle, \mathbf{d})$ is a differential ring (non commutative in general) with $\mathbf{d}(S) = \sum_{w \in X^*} \partial(\langle S|w\rangle)\, w$ (term-by-term differentiation).
2. In the previous sessions (CCRT[16].t, we started from a NCDE, NonCommutative Differential Equation) within $(R\langle\langle X\rangle\rangle, \mathbf{d})$

$$S' = MS \tag{1}$$

   `(+ various conditions)` and discussed many algebraic aspects of a supposed existing solution. Let us remark that a solution of homogeneous equation other than zero need not exist.
3. We will be interested today by analytic aspects i.e. with $\emptyset \neq \Omega \subset \mathbb{C}$ connected.

$$(R, \partial) = (\mathcal{H}(\Omega), \frac{d}{dz}) \tag{2}$$

$S' = MS$ and iterated integrals/2

4. We remark at once that, in our context of today i.e. (2), the derivative ∂ admits a section and this will give rise to a way of solving **all** NCDE with $M \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$. The method is the following
   1. Pick a $z_0 \in \Omega$ and then form the sequence in $\mathcal{H}(\Omega)$
   $$S_0 = 1_{X^*}\ ;\ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n(s)ds \tag{3}$$
   2. This sequence converges in $\mathcal{H}(\Omega)$, for the topology of stationary convergence (TSCV), to
   $$S_{Pic}^{z_0} := \lim_{n\to\infty} S_n \tag{4}$$
   (Picard's process)
   3. The set of solutions of (1) in our context (2) is then $S_{Pic}^{z_0}.\mathbb{C}\langle\langle X\rangle\rangle$.
   4. Remark that, in the context of iterated integrals,
   $$S_{Pic}^{z_0} = \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\, w.$$

$S' = MS$ and iterated integrals/3

5. (Iterated integrals) The context of iterated integrals corresponds to the case when the multiplier is homogeneous of degree one (i.e. M is of the form $\sum_{x\in X} u_x\, x$) and an initial condition $S(z_0) = 1_{\mathbb{C}\langle\langle X\rangle\rangle}$. All together, we get the system
$$S' = MS\ ;\ S(z_0) = 1_{\mathbb{C}\langle\langle X\rangle\rangle} \tag{5}$$
the (unique) solution $S_{Pic}^{z_0}$ is obtained by iterated integrals (we supposed Ω to be connected and now simply connected). Explicitely put with $w = x_1 \dots x_n$
$$\alpha_{z_0}^{z}(w) = \int_{z_0}^{z} u_{x_1}\, ds_1 \int_{z_0}^{s_1} \dots \Big[\int_{z_0}^{s_j} u_{x_{j+1}}\, ds_{j+1}\Big] \dots \int_{z_0}^{s_{n-1}} u_{x_n}\, ds_n \tag{6}$$

6. Summarizing: with the above and $S = \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\, w (= \alpha_{z_0}^{z})$ one gets the solution of
$$S' = \Big(\sum_{x\in X} u_x\, x\Big) S\ ;\ S(z_0) = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \tag{7}$$

$S' = MS$ and iterated integrals/4

7. **Remark.** – For a solution of, only

$$S' = MS,\ \underbrace{\langle S|1_{X^*}\rangle = 1_\Omega}_{Magnus}$$

no need to use the same lower bound z_0, one has only to take antiderivatives. So, one could as well have a collection $(z_w)_{w\in X^*}$ of lower bounds taken in Ω or at its frontier, providing that the so generated improper integrals converged.

8. This is the case, in particular for (5), in the doubly-cleft plane, fuschian inputs[a] and where the condition $S(z_0) = 1_{\mathbb{C}\langle\langle X\rangle\rangle}$ is replaced by $\lim_{z\to 0} S(z)e^{-x_0\log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle}$, namely

$$S' = MS\ ;\ \lim_{z\to 0} S(z)e^{-x_0\log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \tag{8}$$

[a] $\Omega = \mathbb{C}\setminus(]-\infty,0]\cup[1,+\infty[),\ u_0 = 1/z,\ u_1 = 1/(1-z)$.

$S' = MS$ and iterated integrals/5

9. Solution of NCDE with asymptotic condition (8) may not exist, but if it does, it is unique.
10. For example the above (8) has a unique solution (Drinfel'd G_0, see [11]) and it can be shown that we can replace $\lim_{z\to 0} S(z)e^{-x_0 \log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle}$ by any condition of the form $\lim_{z\to 0} S(z)e^{-x_0 \log(z)} U = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle}$ with success (i.e. getting a solution, but not necessarily the same) and $U \in \mathbb{C}\langle\langle X\rangle\rangle$ iff

$$U \in \underbrace{1 + \mathbb{C}_+\langle\langle X\rangle\rangle}_{Mag(\mathbb{C},X)}$$

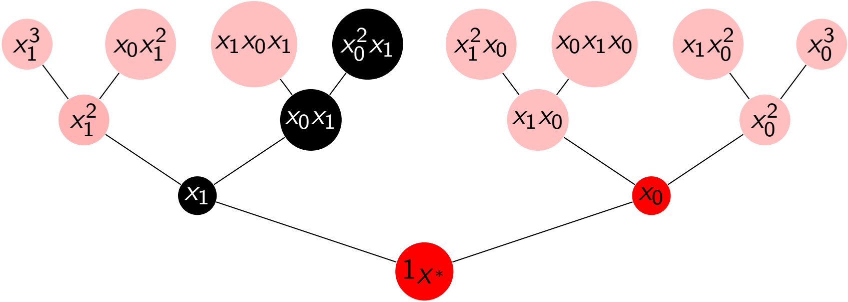
Explicit construction of Drinfeld's G_0

11. Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \displaystyle\int_0^z \alpha_0^s(u)\frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \displaystyle\int_1^z \alpha_1^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \displaystyle\int_0^z \alpha_0^s(u)\frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \end{cases}$$

12. One can show (left to the reader) that $\sum_{w \in X^*} \alpha_0^z(w)\, w$ is precisely Drinfeld's G_0.

The tree of iterated integrals



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$. Jonquière branch ($= x_0^* x_1$) is in black.

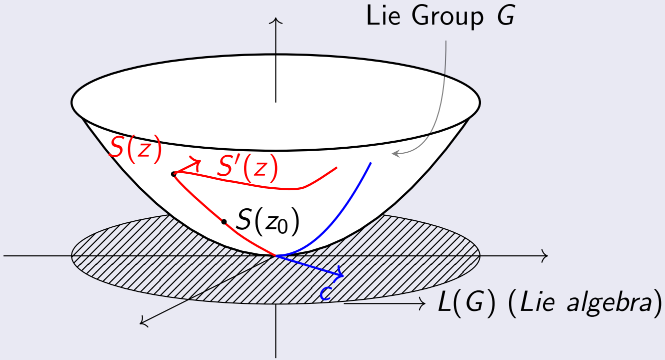
$$\langle S|x_0^n\rangle = \frac{\log(z)^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_0^{k-1}x_1\rangle = \underbrace{\mathrm{Li}_k(z)}_{Jonquière} = \mathrm{Li}_{x_0^{k-1}x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^k}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1^2 n_2} \quad ; \quad \langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!}$$

Solutions as paths drawn on the Magnus group.

13. The paradigm we will use in the future is that, if $S(z)$ (each coordinate holomorphic), drawn on the Magnus group is such that
    1. $S(z_0)$ belongs to some closed subgroup G
    2. $\mathbf{d}(S)S^{-1}[z] = M(z)$ belongs, for all $z \in \Omega$ to the tangent space $T_1(G)$.
    3. Here $S(z_0)$ is replaced by a limit condition (as if $z_0 \in \overline{\Omega}$) we will exploit the subgroup (i.e. Hausdorff) algebraically.



Main difference between $\alpha_{z_0}^z$ and α_0^z.

14. Here, we still work with
$\Omega = \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and $u_0 = 1/z,\ u_1 = 1/(1-z)$
15. $\alpha_{z_0}^z,\ \alpha_0^z :\ X^* \longrightarrow \mathcal{H}(\Omega)$ are both shuffle characters (see below) but they satisfy different growth conditions.
16. **With $\alpha_{z_0}^z$, ($z_0 \in \Omega$). –** Let us denote $\mathfrak{K}(\Omega)$ the set of compact subsets of Ω. One can show that, for all $K \in \mathfrak{K}(\Omega)$, there exists $M_K > 0$ s.t.
$$(\forall w \in X^+)(\ ||\langle \alpha_{z_0}^z | w \rangle||_K \leq M_K \frac{1}{(|w|-1)!}\ ) \tag{9}$$
17. This entails that, given a rational series $T = \sum_{n \geq 0} T_n$ (where $T_n = \sum_{|w|=n} \langle T | w \rangle$), the series, for all $K \in \mathfrak{K}(\Omega)$
$$\sum_{n \geq 0} ||\langle \alpha_{z_0}^z | T_n \rangle||_K < +\infty$$
18. We will say that $T \in Dom(\alpha_{z_0}^z)$ and set $\alpha_{z_0}^z(T) = \sum_{n \geq 0} \langle \alpha_{z_0}^z | T_n \rangle$.

Main difference between $\alpha_{z_0}^z$ and $\alpha_0^z/2$

19. In fact, α_0^z satisfies no condition of the type (9) because, with $x_0^*x_1$ (Jonquière branch), we can see that

    1. for $n \geq 1$, $(x_0^*x_1)_n = x_0^{n-1}x_1$, then

    $$\langle \mathrm{Li}(z)|x_0^{n-1}x_1\rangle = \langle \alpha_{z_0}^z|x_0^{n-1}x_1\rangle = J_n(z) = \sum_{k\geq 1}\frac{z^k}{k^n} \tag{10}$$

    2. The series $\sum_{n\geq 0} J_n$ does not converge (even pointwise) on $]0,1[$ because,

    $$x \in]0,1[\Longrightarrow J_n(x) \geq x$$

    3. So, what can be salvaged ? $\rightarrow$ in fact, conditions (growth or other) implying absolute convergence at the level of words is hopeless because of restriction and we would like to preserve

    $$\mathrm{Li}(x_0^*) = z\ ;\ \mathrm{Li}(x_1^*) = 1/(1-z)\ ;\ \mathrm{Li}(S \sqcup\!\sqcup T) = \mathrm{Li}(S).\mathrm{Li}(T) \tag{11}$$

    and then $\mathrm{Li}\left((x_0+x_1)^*\right) = z/(1-z)$

Main difference between $\alpha_{z_0}^z$ and $\alpha_0^z/3$

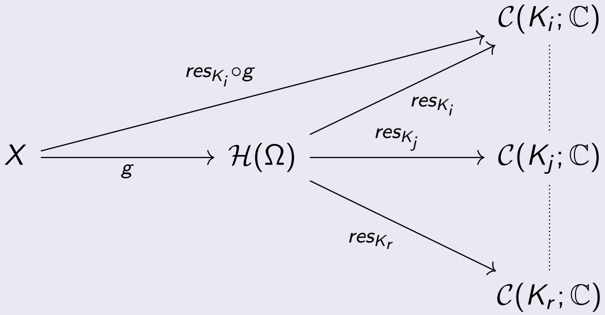
20. Then, we must have a criterium (for admitting a series in $Dom(\mathrm{Li})$)
21. Fortunately $\mathcal{H}(\Omega)$ shares with finite dimensional spaces the following property

$$\text{Unconditional convergence} \Longleftrightarrow \text{Absolute convergence} \qquad (12)$$

22. **Unconditional convergence** for a series $\sum_{n\geq 0} u_n$ means convergence "independent of the order" i.e. that $\sum_{n\geq 0} u_{\sigma(n)}$ converges whatever $\sigma \in \mathfrak{S}_{\mathbb{N}}$.
23. **Absolute convergence** is related to the seminorms of the space.
24. Time is ripe now to speak of the standard topology of $\mathcal{H}(\Omega)$.
25. For $K \in \mathfrak{K}(\Omega)$, we introduce the seminorm (norm if Ω is connected and $K^\circ \neq \emptyset$)

$$||f||_K = \sup_{z\in K} |f(z)|$$

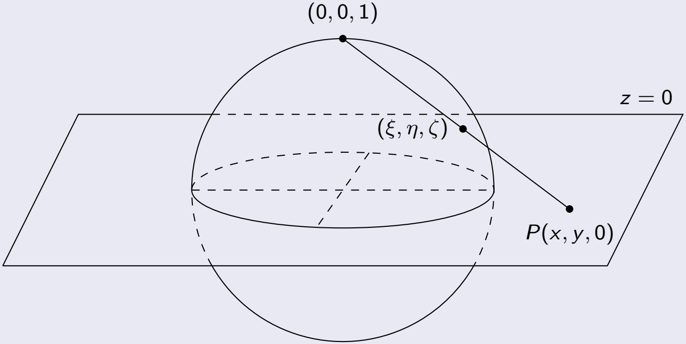
26 We now use a very very general construction, well suited both for series and holomorphic functions (and many other situations), that of initial topologies (see [38] and, for a detailed construction [4], Ch1 §2.3)



27 So $\mathcal{H}(\Omega)$ is a locally convex TVS whose topology is defined by the family of seminorms $(|| \, ||_K)_{K\in\mathfrak{K}(\Omega)}$.

Topology of $\mathcal{H}(\Omega)$ cont'd.

26. In fact, every $\Omega \subset \mathbb{C}$ is σ-compact, this means that one can construct a sequence $(K_n)_{n\in\mathbb{N}}$ of compacts i.e. $(\forall K \in \mathfrak{K}(\Omega))(\exists n \in \mathbb{N})(K \subset K_n)$ therefore $\mathcal{H}(\Omega)$ is a complete (hence closed) subset of the product $\Pi_{n\in\mathbb{N}}\,\mathcal{C}(K_n;\mathbb{C})$. To see this, add a point at infinity to the plane $\mathbb{C}$ and consider the stereographic projection.



Properties of $\mathcal{H}(\Omega)$ and domain of Li.

26. If $\Omega \neq \emptyset$, $\mathcal{H}(\Omega)$ is not normable because, there are two continuous operators

$$a^\dagger:\ f \mapsto z.f\ ;\ a:\ f \mapsto \frac{d}{dz}f$$

such that $[a, a^\dagger] = Id_{\mathcal{H}(\Omega)}$ (**Hint** Compute $ad_a(e^{ta^\dagger})$).

27. $\mathcal{H}(\Omega)$ has property (12).

28. This leads us to the following

Definition

Let $T \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$, we define (with $[S]_n := \sum_{|w|=n}\langle S|w\rangle\, w$)

$$Dom(T) = \{S \in \mathbb{C}\langle\langle X\rangle\rangle \mid \sum_{n\geq 0}\langle T|\ [S]_n\rangle \text{ cv inconditionally}\} \tag{13}$$

If $S \in Dom(T)$, we set $\langle T|S\rangle := \sum_{n\geq 0}\langle T|\ [S]_n\rangle$.

Shuffle properties and domain of Li.

26. In the case when T is a shuffle character, we have

Theorem (GD, Quoc Huan Ngô, HNM [16] for Li)

Let $T \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ such that

$$\langle T \mid :\ P \mapsto \langle T|P\rangle \ (\mathbb{C}\langle X\rangle \to \mathcal{H}(\Omega)) \tag{14}$$

is a shuffle character. then
i) Dom(T) is a shuffle subalgebra of $(\mathbb{C}\langle\langle X\rangle\rangle, ш, 1_{X^*})$.
ii) $\langle T|S_1 ш S_2\rangle = \langle T|S_1\rangle\langle T|S_2\rangle$ i.e. $S \mapsto \langle T|S\rangle$ is a shuffle character of $(Dom(T), ш, 1_{X^*})$ that we will still denote $\langle T \mid$.
iii) Then $Im(\langle T \mid)$ is a (unital) subalgebra of $\mathcal{H}(\Omega)$.
iv) In particular (see **infra** *for an algebraic proof), $z = \mathrm{Li}(x_0^*)$ and then, $\mathbb{C}[z] \subset Im(\mathrm{Li})$.*

Open problems and some solved

27. Do we have $\mathcal{H}(\Omega) = \overline{Im(Dom(\mathrm{Li}))}$ $(= \overline{Im(\mathrm{Li})})$? (in other words does it exist inaccessible $f \in \mathcal{H}(\Omega)$?)
28. If $z_0 \notin \Omega$, does $1/(z - z_0)$ belong to $Im(\mathrm{Li})$? ($z_0 \in \overline{\Omega}$ and $z_0 \notin \overline{\Omega}$)
29. (Solved) Are there non-rational series in $Dom(\mathrm{Li})$? (answer **yes**)
30. (Solved) Is $\mathbb{C}^{rat}\langle\langle X \rangle\rangle$ contained in $Dom(\mathrm{Li})$ (answer **no**)
31. What is the topological complexity of $Dom(\mathrm{Li})$ in the **Borel hierarchy** (Addison notations, see [22] for details and use the convenient framework of polish spaces [5], ch IX).
32. **Borel hierarchy**: We recall that this hierarchy is indexed by ordinals and defined as follows
    1. A set is in $\mathbf{\Sigma}^0_1$ if and only if it is open.
    2. A set is in $\mathbf{\Pi}^0_\alpha$ if and only if its complement is in $\mathbf{\Sigma}^0_\alpha$.
    3. A set A is in $\mathbf{\Sigma}^0_\alpha$ for $\alpha > 1$ if and only if there is a sequence of sets $A_1, A_2, \ldots$ such that each A_i is in $\mathbf{\Pi}^0_{\alpha_i}$ for some $\alpha_i < \alpha$ and $A = \bigcup A_i$.
    4. A set is in $\mathbf{\Delta}^0_\alpha$ if and only if it is both in $\mathbf{\Sigma}^0_\alpha$ and in $\mathbf{\Pi}^0_\alpha$.

Open problems and some solved/2

33. From slide (10), one can remark that the iterated integrals are based on two integrators, informally defined as

$$\iota_1(f) := \int_0^z f(s)\frac{ds}{1-s}\ ;\ \iota_0(f) := \int_{z_0}^z f(s)\frac{ds}{s} \text{ with } z_0 \in \{0,1\} \qquad (15)$$

ι_1 is defined and continous on $\mathcal{H}(\Omega)$ and ι_0 is defined on $span_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}$[a] (context-dependent) and not continuous [16] on this set (see below).
Problem What is the Baire class of ι_0 ?

34. Recall that $\mathfrak{K}(\Omega)$ admits a cofinal sequence $(K_n)_{n\in\mathbb{N}}$ of compacts i.e. $(\forall K \in \mathfrak{K}(\Omega))(\exists n \in \mathbb{N})(K \subset K_n)$ therefore $\mathcal{H}(\Omega)$ is a complete (hence closed) subset of the product $\Pi_{n\in\mathbb{N}}\mathcal{C}(K_n;\mathbb{C})$.

35. An alternative way (see [16]) is to define

$$K_n = \{z \in \Omega \mid d(z,z_0) \leq n \text{ and } d(z,\mathbb{C}\setminus\Omega) \geq \frac{1}{n}\}.$$

[a] It can be a little bit extended, see our paper [16].

Discontinuity of ι_0.

To show discontinuity of ι_0, one of the possibilities consists in exhibiting two sequences $f_n, g_n \in \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$ converging to the same limit but such that

$$\lim \iota_0(f_n) \neq \lim \iota_0(g_n).$$

Here, we choose the function z for being approached in a twofold way and if ι_0 were continuous, we would have equality of the limits of the image-sequences (and this is not the case). We first remark that

$$z = \sum_{n\geq 0} \frac{\log^n(z)}{n!} = \sum_{n\geq 1} (-1)^{n+1} \frac{\log^n((1-z)^{-1})}{n!} = \mathrm{Li}(x_0^*) = 1 - \mathrm{Li}((-x_1)^*)$$

Set

$$f_n = \sum_{0\leq m\leq n} \frac{\log^m(z)}{m!} \quad \text{and} \quad g_n = \sum_{1\leq m\leq n} (-1)^{m+1} \frac{\log^m((1-z)^{-1}))}{m!}$$

(these two sequences are in $\mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*}$).

Discontinuity of ι_0. cont'd

It is easily seen that $\iota_0(f_n) = f_{n+1} - 1$ and then that $\lim\limits_{n\to+\infty} \iota_0(f_n)(z) = z - 1$.
Now, for any $s \in [0, z]$ with $z \in]0, 1[$, one has

$$|g_n(s)| = |\sum_{m=1}^{n} (-1)^{m+1} \frac{\log^n(1-s)}{m!}| \leq \sum_{m=1}^{n} \frac{|\log^n(1-s)|}{m!}$$
$$\leq \exp(-\log(1-s)) - 1 = \frac{s}{1-s}.$$

In order to exchange limits, we apply *Lebesgue's dominated convergence theorem* to the measure space $(]0, z], \mathcal{B}, dz/z)$ ($\mathcal{B}$ is the usual Borel σ-algebra) and the function $p(x) = s(1-s)^{-1}$ which is - as are the functions g_n - integrable on $]0, z]$ (for every $z \in]0, 1[$). Then

$$\lim(\iota_0(g_n)) = \lim_{n\to+\infty} \int_0^z g_n(s) \frac{ds}{s} = \int_0^z \lim_{n\to+\infty} g_n(s) \frac{ds}{s} = \int_0^z s \frac{ds}{s} = z.$$

Hence, for $z \in]0, 1[$, we obtain, $\lim(\iota_0(f_n)) = z - 1 \neq z = \lim(\iota_0(g_n))$ This completes the proof.

Li as a shuffle character (Lie theoretical proof, sketched).

36. Recall what has been said in one of our previous CCRT about the Hausdorff group of the Hopf algebra $(\mathbb{C}\langle X\rangle, ш, 1_{X^*}, \Delta_{\mathtt{conc}}, \epsilon)$ (the antipode exists but is not needed here). Let us recall its features
    1. The shuffle product between two words is defined by recursion or duality (see our paper [13])
    2. $\Delta_{\mathtt{conc}}$, the dual of conc is defined, within $\mathbb{C}\langle X\rangle$, by duality
    $$\langle \Delta_{\mathtt{conc}}(w) | u \otimes v\rangle = \langle w | uv\rangle$$
    or combinatorially $\Delta_{\mathtt{conc}}(w) = \sum_{uv=w} u \otimes v$
    3. $\epsilon(P) = \langle P | 1_{X^*}\rangle$
37. For every Hopf algebra $(\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$, the set $\Xi(\mathcal{B})$ of characters of $(\mathcal{B}, \mu, 1_{\mathcal{B}})$ is a group under convolution (a monoid in case of a general bialgebra, see our paper [14] Prop. 5.6).
38. Here, due to the fact that $\mathbb{C}$ is a field, we can characterize the group of shuffle characters $\Xi(\mathcal{B})$ by the (algebraic) equations

$$\langle S | 1_{X^*}\rangle = 1_{\mathbb{C}} \; ; \; \Delta_{ш}(S) = S \otimes S \tag{16}$$

Li as a shuffle character/2

39 Let us now consider an evolution equation $S' = M.S$ in $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ with a primitive multiplier i.e., for all $z \in \Omega$,

$$\Delta_{ш}(M(z)) = M(z) \otimes 1_{X^*} + 1_{X^*} \otimes M(z)$$

40 Then, if S is group-like (for $\Delta_{ш}$) at one point $z_0 \in \Omega$, it is group-like everywhere (we will see that the point can be remote, or frontier).

41 Let us have a look at the proof, from which we will deduce the version with asymptotic initial condition. We propose the first following statement

Proposition

Let be given, within $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$, the following evolution equation

$$S' = M.S \; ; \; S(z_0) = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \tag{17}$$

we suppose that, for all $z \in \Omega$, $M(z)$ is primitive (for $\Delta_{ш}$).

Then, *for all $z \in \Omega$, $S(z)$ is group-like (for $\Delta_{ш}$). This means that S is a character of $(\mathcal{H}(\Omega)\langle X\rangle, ш, 1_{X^*})$.*

Li as a shuffle character/3

Proof

42. Firstly, we transform (17) by $\Delta_{\text{ш}}$ (which commute - easy exercise - with derivation)

$$\Delta_{\text{ш}}(S)' = \Delta_{\text{ш}}(S') = \Delta_{\text{ш}}(M).\Delta_{\text{ш}}(S)\ ;\ \Delta_{\text{ш}}(S(z_0)) = 1 \otimes 1$$

43. Taking into account that M is primitive, we get

$$\Delta_{\text{ш}}(S)' = (M \otimes 1 + 1 \otimes M).\Delta_{\text{ш}}(S)\ ;\ \Delta_{\text{ш}}(S(z_0)) = 1 \otimes 1 \quad (18)$$

44. Let us see what happens to $S \otimes S$

$$(S \otimes S)' \overset{(1)}{=} S' \otimes S + S \otimes S' = MS \otimes S + S \otimes MS = (M \otimes 1 + 1 \otimes M).(S \otimes S) \quad (19)$$

45. We see that $\Delta_{\text{ш}}(S)$ and $S \otimes S$ satisfy the same evolution equation (same multiplier) and same initial condition (at z_0).

Li as a shuffle character/4

Proof

46. Then, for every $z \in \Omega$, we have $\Delta_{ш}(S(z)) = S(z) \otimes S(z)$ (and still $\langle S(z)|1_{X^*}\rangle = 1_{\mathbb{C}}$).
47. Finally, as $S(z)$ is a character for every $z \in \Omega$, we get that S is a character of $(\mathcal{H}(\Omega)\langle X\rangle, ш, 1_{X^*})$.

Let us try this one.

48. As an excellent exploratory exercise, we can try the multiplier
$$u_0.x_0 + u_1.x_1 + u_2.[x_0, x_1]$$
with $u_i \in \mathcal{H}(\Omega)$.
49. For example, with
$u_0 = 1/z,\ u_1 = 1/(1-z),\ u_2 = \big(2\,\mathrm{Li}_2 + \log(z)\log(1-z)\big)'$ we do not have linear independence of $(\langle S|w\rangle)_{w\in X^*}$.

What is the condition ? (Forthcoming talk)

Some shuffle subalegbras of $Im(\mathrm{Li}_\bullet)$ and their images.

50. Starting point $(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*})$

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, \shuffle, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle \shuffle \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \shuffle \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\uparrow & \nearrow & \\
\mathbb{C}\langle X\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \otimes_{\mathbb{C}} \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & &
\end{array}$$

51. These extensions, as well as closed subgroup properties will be the subject of forthcoming talks.

Concluding remarks

- We have started with iterated integrals (trees, $S' = MS$, primitives, sectioned subalgebras towards integro-differential rings.)
- Generating series of iterated integrals satisfy a very special class of NCDE $S' = MS$ (i.e. with multiplier of the type $M = \sum_{x \in X} u_x\, x$ and initial condition $S(z_0) = 1$).
- This entails that the solution of (NCDE + Init) is a shuffle character.
- Other solutions with then same multiplier share this property (shuffle character), i.e. the solutions with asymptotic initial condition.
- In particular the arrow Li $(Dom(\mathrm{Li}))$
- Integrators ι_i (discontinuity of ι_0)
- Discussion, open questions
  1. Topological complexity of $Dom(\mathrm{Li}), \mathrm{Li}(Dom(\mathrm{Li}))$
  2. Baire class of ι_0

THANK YOU FOR YOUR ATTENTION !

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